AKHIMO 2023 Problems

Problem 1. Consider the following functions $f, g : (0,1) \to \mathbf{R}$ given by

$$f(x) = \frac{x}{1-x^2} \quad \text{and} \quad g(x) = \frac{x^{2023}}{1-x^2}.$$

Compare the following numbers: $f^{(2022)}\left(\frac{1}{2023}\right)$ and $g^{(2022)}\left(\frac{1}{2023}\right)$, where $f^{(n)}$ denotes the n^{th} derivative of the function f.

Problem 2. Let $\{x_n\}$ be a sequence defined as follows:

$$x_1 \in \left[0, \frac{\pi}{2}\right] \quad \text{and} \quad x_{n+1} = \sqrt[3]{\frac{\sin^3 x_1 + \cdots + \sin^3 x_n}{n}} \quad \text{for} \quad n \geq 1.$$

Find the limit $\lim\limits_{n\to\infty} x_n^3 \ln n$.

Problem 3. Let $A, B, C \in \mathcal{M}_n(\mathbf{R})$ be skew-symmetric $(M^T = -M)$ matrices such that:

$$\det(A^2 + B^2 + C^2) = 0.$$

Prove that for each triple $P, Q, R \in \mathcal{M}_n(\mathbf{R})$ the following equality holds

$$\det(AP + BQ + CR) = 0.$$

Problem 4. Let $f : [0,1] \to [0,1]$ be a continuous function with a fixed point $\xi \in (0,1)$. Assume that f is differentiable at ξ and that $f'(\xi) < -1$. Prove that there exist $x_0, x_1, 0 \leq x_0 < \xi < x_1 \leq 1$, such that $f(x_0) = x_1$ and $f(x_1) = x_0$.

Problem 5. Let $A = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be a diagonal $n \times n$ matrix, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are real numbers. Prove that there exists a unitary matrix U and real number λ such that

$$|AU - UA| \geq |A - \lambda E|,$$

where $|B| = \sqrt{B^* B}$ square root, E is the identity matrix, and $X \geq Y$ means that the difference $X - Y$ is a positive definite matrix.